J. Fluid Mech. (1978), *vol.* 86, *part* 3, *pp.* 415–431
Printed in Great Britain

Long nonlinear internal waves in channels of arbitrary cross-section

By R. H. J. GRIMSHAW
Department of Mathematics, University of Melbourne, Parkville, Victoria 3052, Australia

(Received 3 May 1977 and in revised form 10 August 1977)

Using a two-layer fluid model, equations are developed which describe long internal waves propagating along a channel of arbitrary cross-section. Expressions for the phase speed of these waves are derived in terms of geometric properties of the cross-section. When weakly nonlinear effects are balanced by weak dispersion, a Korteweg–de Vries equation is derived to describe the waves. The effects of a slowly varying cross-section are included. Applications of the theory are made to recent observations of internal surges in lakes.

1. Introduction

In recent years there have been a number of observations of long internal waves occurring on the thermocline in inland lakes and in coastal waters. All the observations show the formation of an internal surge, which steepens owing to nonlinear effects and subsequently evolves into a train of shorter period waves, which have tentatively been identified as solitary waves, or solitons. These internal surges have been recorded in the Strait of Gibraltar (Ziegenbein 1969), Massachusetts Bay (Halpern 1971; Lee & Beardsley 1974) and at Scripps Pier in La Jolla, California (Winant 1974). Recent measurements in inland lakes have been made in Loch Ness (Thorpe 1971; Thorpe, Hall & Crofts 1972), Seneca Lake in New York State (Hunkins & Fliegel 1973) and Lake Babine in British Columbia (Farmer 1978). At present the method of generation of these surges is obscure and there may well be no unique mechanism. However, after the generation period, an appropriate equation with which to describe the waves is an equation of Korteweg–de Vries type (Hunkins & Fliegel 1973; Lee & Beardsley 1974). An apparent exception is Loch Ness, where the surge reflects strongly off the ends of the lake and may sometimes be in partial resonance with the wind (Thorpe 1974).

The theories used so far to describe internal surges have used a two-layer fluid model, and have ignored both topographic effects in the direction transverse to the surge propagation and changes in topography in the direction of surge propagation. Farmer (1978), in particular, has drawn attention to the possible importance of the latter effect. The purpose of this paper is to provide a theory which takes account of both these effects. We shall use a two-layer fluid model and consider weakly nonlinear internal waves in a channel of arbitrary cross-section. In § 2 we shall derive an expression for the speed of linear long waves, based on the assumption that the dimensional wavelength $\bar{\lambda}$ is much greater than $\bar{h}$, which is a suitable linear dimension associated with the channel cross-section (e.g. the cross-sectional area divided by the width of the

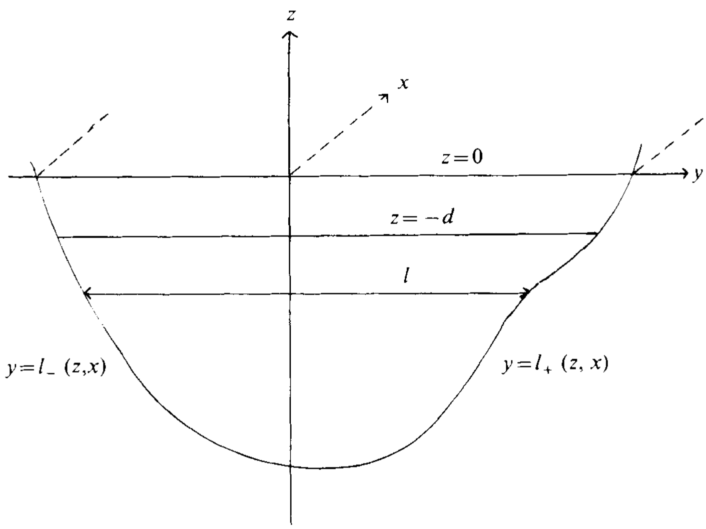

FIGURE 1. The co-ordinate system.

interface between the two fluids). Then in §3 we shall consider weakly nonlinear long waves and derive a Korteweg–de Vries equation when the nonlinear effects are balanced by dispersive effects; if $\bar{a}$ is the dimensional wave amplitude, then the measure of nonlinearity is $\bar{a}\,\bar{h}^{-1}$ and we shall require this small parameter to be comparable with $\bar{h}^2\bar{\lambda}^{-2}$, which is the appropriate measure of dispersion. If the interface is shallow, then this criterion must be re-examined and we show in §4 that the measure for nonlinearity is then $\bar{a}\,\bar{d}^{-1}$, while that for dispersion is $\bar{d}\bar{h}\bar{\lambda}^{-2}$, where $\bar{d}$ is the depth of the interface. We shall also allow the cross-section to vary slowly in the direction of wave propagation; the relative measure for this slow variation is ϵ^3, where $\epsilon^2 = \bar{a}\,\bar{h}^{-1}$. The Korteweg–de Vries equation that we shall derive is the counterpart for internal waves of the equation derived by Peregrine (1968) for surface waves in a channel of uniform cross-section. Shen (1968) considered the propagation of weakly nonlinear long waves in a compressible fluid down a channel of arbitrary cross-section and obtained a Korteweg–de Vries equation for the variation in wave amplitude; subsequently Shen & Keller (1973) extended this work to include the effects of rotation and of a slow variation in the channel cross-section. However, their results are not directly applicable to the present problem, as they assumed that the fluid was barotropic, and so were unable to take a limit in which the fluid becomes incompressible yet retains a density stratification. In §4 we shall consider applications. The theory that we shall develop is intended to be applicable mainly to the observations of internal surges made in inland lakes, and particularly to the observations made by Farmer (1978). However the theory may also be relevant to the surges observed in coastal waters when the topography has a strong directional preference (e.g. the Strait of Gibraltar). It should be noted that our theory implicitly assumes that the horizontal dimension of the channel is of the same order of magnitude as the vertical dimension $\bar{h}$. However, this is not as severe a restriction as it might seem on first sight; the crucial assumption is that the wave amplitude (e.g. the height of the interface) should have a greater variation in the direction of wave propagation than in the transverse direction. Clearly this latter condition may well be satisfied even in wide channels (e.g. in a rectangular channel there are no variations in the transverse direction at all).

We shall use non-dimensional co-ordinates based on the length scale $\bar{h}$, the time scale $(\bar{h}g^{-1}\beta^{-1})^{\frac{1}{2}}$ and the velocity scale $(\bar{h}g\beta)^{\frac{1}{2}}$. Here β is the ratio $(\rho_2-\rho_1)(\rho_2+\rho_1)^{-1}$, where $\rho_1\,(\rho_2)$ is the density in the upper (lower) layer. Our scaling is based on the anticipated order of magnitude associated with the internal, or interfacial waves. β is usually a small parameter (for the observed temperature differences of 6–10 °C, β is $O(10^{-3})$), and we shall make extensive use of the Boussinesq limit $\beta\to 0$. The co-ordinate system is described in figure 1. The x co-ordinate varies along the channel; the equilibrium free surface is $z=0$, the equilibrium interface is $z=-d$, and the sides of the channel are specified by $y=l_{\pm}(z,x)$. The width of the channel at the level z is $l=l_+-l_-$, the width of the equilibrium free surface is $l_0=l$ at $z=0$, while the width of the equilibrium interface is $l_d=l$ at $z=-d$. The flow, assumed inviscid and incompressible, is irrotational in each layer. We shall let ϕ_i be the velocity potential in R_i, where R_1 is the upper layer and R_2 is the lower layer. Laplace's equation is satisfied in each region, hence

$$\phi_{ixx}+\phi_{iyy}+\phi_{izz}=0 \quad \text{in} \quad R_i, \quad i=1,2. \tag{1.1}$$

The boundary conditions at the free surface $z=\zeta(x,y,t)$ are

$$\left.\begin{aligned}\zeta_t-\phi_{1z}+\zeta_x\phi_{1x}+\zeta_y\phi_{1y}&=0\\ \zeta+\beta(\phi_{1t}+\tfrac{1}{2}\phi_{1x}^2+\tfrac{1}{2}\phi_{1y}^2+\tfrac{1}{2}\phi_{1z}^2)&=0\end{aligned}\right\} \quad \text{at} \quad z=\zeta. \tag{1.2}$$

The first condition is the kinematic condition, while the second condition is the dynamic condition of constant pressure. The displacement of the interface is $\eta(x,y,t)$, and at $z=-d+\eta$, the boundary conditions are the kinematic condition and the dynamic condition of continuous pressure. Hence

$$\left.\begin{aligned}&\eta_t-\phi_{iz}+\eta_x\phi_{ix}+\eta_y\phi_{iy}=0, \quad i=1,2,\\ &(1-\beta)\{\eta+\beta(\phi_{1t}+\tfrac{1}{2}\phi_{1x}^2+\tfrac{1}{2}\phi_{1y}^2+\tfrac{1}{2}\phi_{1z}^2)\}\\ &\quad=(1+\beta)\{\eta+\beta(\phi_{2t}+\tfrac{1}{2}\phi_{2x}^2+\tfrac{1}{2}\phi_{2y}^2+\tfrac{1}{2}\phi_{2z}^2)\}\end{aligned}\right\} \quad \text{at} \quad z=-d+\eta. \tag{1.3}$$

Both these boundary conditions (1.2) and (1.3) may be linearized about $z=0$ and $z=-d$ respectively. Retaining only the linear and quadratic terms we find that, after some manipulation using (1.1),

$$\left.\begin{aligned}\zeta_t-\phi_{1z}+(\zeta\phi_{1x})_x+(\zeta\phi_{1y})_y+\dots&=0\\ \zeta+\beta\{\phi_{1t}+\zeta\phi_{1tz}+\tfrac{1}{2}\phi_{1x}^2+\tfrac{1}{2}\phi_{1y}^2+\tfrac{1}{2}\phi_{1z}^2\}+\dots&=0\end{aligned}\right\} \quad \text{at} \quad z=0 \tag{1.4}$$

and

$$\left.\begin{aligned}&\eta_t-\phi_{1z}+(\eta\phi_{ix})_x+(\eta\phi_{iy})_y+\dots=0, \quad i=1,2,\\ &2\eta=(1-\beta)(\phi_{1t}+\eta\phi_{1tz}+\tfrac{1}{2}\phi_{1x}^2+\tfrac{1}{2}\phi_{1y}^2+\tfrac{1}{2}\phi_{1z}^2)\\ &\qquad-(1+\beta)(\phi_{2t}+\eta\phi_{2tz}+\tfrac{1}{2}\phi_{2x}^2+\tfrac{1}{2}\phi_{2y}^2+\tfrac{1}{2}\phi_{2z}^2)+\dots\end{aligned}\right\} \quad \text{at} \quad z=-d. \tag{1.5}$$

On the sides the normal derivative of ϕ must vanish:

$$\phi_{iy}=l_{\pm x}\phi_{ix}+l_{\pm z}\phi_{iz} \quad \text{on} \quad y=l_{\pm}(z,x), \quad i=1,2. \tag{1.6}$$

2. Linear long waves in a channel of uniform cross-section

In a channel of uniform cross-section, $l_\pm$ are independent of x, and the boundary condition (1.6) becomes

$$\phi_{iy} = l_{\pm z}\phi_{iz} \quad \text{on} \quad y = l_\pm(z), \quad i = 1, 2. \tag{2.1}$$

The equations governing linear waves are then (1.1) with the boundary condition (2.1) and the linearized forms of the boundary conditions (1.4) and (1.5):

$$\zeta_t = \phi_{1z}, \quad \zeta + \beta\phi_{1t} = 0 \quad \text{on} \quad z = 0, \tag{2.2}$$

$$\eta_t = \phi_{1z} = \phi_{2z}, \quad 2\eta = (1-\beta)\phi_{1t} - (1+\beta)\phi_{2t} \quad \text{on} \quad z = -d. \tag{2.3}$$

We shall seek a solution of (1.1) and (2.1)–(2.3) for which

$$\phi_i = \hat{\phi}_i(y, z)\exp(i\kappa x - i\omega t), \tag{2.4}$$

with similar expressions for η and ζ. Here κ is the wavenumber, ω is the wave frequency and the phase speed is $c = \omega\kappa^{-1}$. On substituting (2.4) into the governing equations, and subsequently simplifying, it follows that

$$\left.\begin{aligned} \hat{\phi}_{iyy} + \hat{\phi}_{izz} &= \kappa^2\hat{\phi}_1 \quad \text{in} \quad R_i, \quad i = 1, 2, \\ \hat{\phi}_{iy} &= l_{\pm z}\hat{\phi}_{iz} \quad \text{on} \quad y = l_\pm(z), \quad i = 1, 2, \\ \hat{\phi}_{1z} &= \beta\kappa^2c^2\hat{\phi}_1 \quad \text{on} \quad z = 0, \\ \hat{\phi}_{1z} = \hat{\phi}_{2z} &= -\tfrac{1}{2}\kappa^2c^2\{(1-\beta)\hat{\phi}_1 - (1+\beta)\hat{\phi}_2\} \quad \text{on} \quad z = -d. \end{aligned}\right\} \tag{2.5}$$

Long waves are obtained in the limit $\kappa \to 0$. Hence we shall attempt to solve (2.5) with expansions of the form

$$\left.\begin{aligned} \hat{\phi}_i &= \hat{\phi}_i^{(0)} + \kappa^2\hat{\phi}_i^{(1)} + \kappa^4\hat{\phi}_i^{(2)} + \ldots, \\ c &= c_0 + \kappa^2 c_1 + \ldots. \end{aligned}\right\} \tag{2.6}$$

At the lowest order, it may easily be shown that $\hat{\phi}_i^{(0)}$ is a constant, A_i. Substitution of (2.6) into (2.5) then shows that at the next order in κ^2

$$\left.\begin{aligned} \hat{\phi}_{iyy}^{(1)} + \hat{\phi}_{izz}^{(1)} &= A_i \quad \text{in} \quad R_i, \quad i = 1, 2, \\ \hat{\phi}_{iy}^{(1)} &= l_{\pm z}\hat{\phi}_{iz}^{(1)} \quad \text{on} \quad y = l_\pm(z), \quad i = 1, 2, \\ \hat{\phi}_{1z}^{(1)} &= \beta c_0^2 A_1 \quad \text{on} \quad z = 0, \\ \hat{\phi}_{1z}^{(1)} = \hat{\phi}_{2z}^{(1)} &= -\tfrac{1}{2}c_0^2\{(1-\beta)A_1 - (1+\beta)A_2\} \quad \text{on} \quad z = -d. \end{aligned}\right\} \tag{2.7}$$

Thus $\hat{\phi}_i^{(1)}$ satisfies a Poisson equation with a Neumann boundary condition. In order for (2.7) to have a unique solution $\hat{\phi}_i^{(1)}$ must satisfy the compatibility condition

$$\iint_{R_i} \{\hat{\phi}_{iyy}^{(1)} + \hat{\phi}_{izz}^{(1)}\}\,dy\,dz = \oint_{\partial R_i} \hat{\phi}_{iy}^{(1)}\,dz - \hat{\phi}_{iz}^{(1)}\,dy, \quad i = 1, 2. \tag{2.8}$$

Substituting (2.7) into (2.8), it follows that, with some rearrangement,

$$\left.\begin{aligned} A_2 S_2 &= -A_1(S_1 - \beta c_0^2 l_0), \\ A_2 S_2 &= -\tfrac{1}{2}c_0^2 l_d\{(1-\beta)A_1 - (1+\beta)A_2\}. \end{aligned}\right\} \tag{2.9}$$

Here we recall that S_i is the cross-sectional area of R_i, l_0 is the breadth of the free surface and l_d is the breadth of the interface. Equations (2.9) consist of two linear homogeneous equations for A_1 and A_2, and can have a solution only if the determinant is zero. This condition implies that

$$c_0^4\beta(1+\beta)\,l_0 l_d - c_0^2\{l_d S_1(1+\beta)+l_d S_2(1-\beta)+2\beta l_0 S_2\}+2S_1S_2 = 0. \tag{2.10}$$

This is a quadratic equation for c_0^2 which has two real solutions; the greater of the two solutions corresponds to a surface wave, while the smaller solution corresponds to an interfacial wave. Finally the solution for $\hat{\phi}_i^{(1)}$ may be put in the form

$$\hat{\phi}_i^{(1)} = A_i\psi_i + B_i, \quad i = 1,2, \tag{2.11}$$

where B_i is an undetermined constant and

$$\left.\begin{array}{c} \psi_{iyy}+\psi_{izz} = 1 \quad \text{in} \quad R_i, \quad i = 1,2, \\ \psi_{iy} = l_{\pm z}\psi_{iz} \quad \text{on} \quad y = l_\pm(z), \quad i = 1,2, \\ \psi_{1z} = \beta c_0^2 \quad \text{on} \quad z = 0, \\ \psi_{1z} = -(S_1-\beta c_0^2 l_0)/l_d, \quad \psi_{2z} = S_2/l_d \quad \text{on} \quad z = -d. \end{array}\right\} \tag{2.12}$$

Since B_i is undetermined at this stage, it is convenient to impose the following extra condition on ψ_i:

$$\iint_{R_i} \psi_i\, dy\, dz = 0. \tag{2.13}$$

As we commented in §1, the validity of our procedure requires that ϕ_{iy} (and ϕ_{iz}) be much smaller than ϕ_{ix}; in particular this means that ψ_{iy} (and ψ_{iz}) is $O(1)$ with respect to κ. Thus our results may be applied to wide channels (i.e. when l_d is large) provided that ψ_{iy} (and ψ_{iz}) is $O(1)$.

At the second order in κ^2, we find that

$$\left.\begin{array}{c} \hat{\phi}_{iyy}^{(2)}+\hat{\phi}_{izz}^{(2)} = \phi_i^{(1)} \quad \text{in} \quad R_i, \quad i = 1,2, \\ \hat{\phi}_{iy}^{(2)} = l_{\pm z}\hat{\phi}_{iz}^{(2)} \quad \text{on} \quad y = l_\pm(z), \quad i = 1,2, \\ \hat{\phi}_{1z}^{(2)} = \beta c_0^2\hat{\phi}_1^{(1)}+2\beta c_0 c_1 A_1 \quad \text{on} \quad z = 0, \\ \hat{\phi}_{1z}^{(2)} = \hat{\phi}_{1z}^{(2)} = -\tfrac{1}{2}c_0^2\{(1-\beta)\hat{\phi}_1^{(1)}-(1+\beta)\hat{\phi}_2^{(1)}\}-c_0c_1\{(1-\beta)A_1-(1+\beta)A_2\} \quad \text{on} \quad z = -d. \end{array}\right\} \tag{2.14}$$

In order for (2.14) to have a unique solution $\hat{\phi}_i^{(2)}$ must satisfy the compatibility condition

$$\iint_{R_i}\{\hat{\phi}_{iyy}^{(2)}+\hat{\phi}_{izz}^{(2)}\}\,dy\,dz = \oint_{\partial R_i}\hat{\phi}_{iy}^{(2)}\,dz-\hat{\phi}_{iz}^{(1)}\,dy, \quad i = 1,2. \tag{2.15}$$

Substituting (2.14) into (2.15) and using (2.11), it may be shown that

$$B_2S_2+B_1(S_1-\beta c_0^2 l_0) = 2\beta c_0 c_1 l_0 A_1+\beta c_0^2 A_1\int_{z=0}\psi_1\,dy, \tag{2.16a}$$

$$B_2S_2+\tfrac{1}{2}c_0^2 l_d\{(1-\beta)B_1-(1+\beta)B_2\} = c_0c_1l_d\{(1+\beta)A_2-(1-\beta)A_1\} + \tfrac{1}{2}c_0^2\int_{z=-d}\{(1+\beta)A_2\psi_2-(1-\beta)A_1\psi_1\}\,dy. \tag{2.16b}$$

These linear inhomogeneous equations for B_1 and B_2 have a zero determinant [namely (2.10)], hence there can be a solution only if the inhomogeneous terms are suitably constrained; this constraint determines c_1. It may be shown that

$$\frac{c_1}{c_0}\{2S_1S_2-\beta(1+\beta)c_0^4l_0l_d\} = -\{S_2-\tfrac{1}{2}(1+\beta)c_0^2l_d\}\beta c_0^2\int_{z=0}\psi_1\,dy$$
$$-\tfrac{1}{2}c_0^2\int_{z=-d}\{(1-\beta)S_2\psi_1+(1+\beta)(S_1-\beta c_0^2l_0)\psi_2\}\,dy. \quad (2.17)$$

In the appendix it is shown that c_0c_1 is always negative.

Equation (2.10) determines the long-wave phase speed c_0 explicitly, while (2.17) determines c_1 once ψ_i has been found from (2.12). In the Boussinesq limit $\beta\rightarrow 0$, the phase speed for the interfacial wave is given by

$$c_0^2 = \frac{2S_1S_2}{l_d(S_1+S_2)}\left\{1-\frac{\beta(S_1^2+S_2^2(2l_0l_d^{-1}-1))}{(S_1+S_2)^2}+O(\beta^2)\right\}. \quad (2.18)$$

Since usually $l_0 \geqslant l_d$, the coefficient of the term in β is negative and the effect of a finite value of β is to decrease the phase speed. The phase speed of the surface wave is given by

$$\beta c_0^2 = \frac{S_1+S_2}{l_0}\{1+O(\beta)\}. \quad (2.19)$$

The factor β occurs in the left-hand side of this equation as our scaling of the time was based on the speeds associated with the interfacial wave rather than on those associated with the surface wave. The leading term in (2.19) is identical with the phase speed for surface waves on a homogeneous fluid (Lamb 1932, §169). However, taking the limit $\beta\rightarrow 0$ in (2.17), using (2.14), does not produce the expression for c_1 which was obtained by Peregrine (1968) for surface waves on a homogeneous fluid. The reason is that for the *surface wave* the limits $\beta\rightarrow 0$ and $\kappa^2\rightarrow 0$ do not commute in the present formulation; it may be shown that an alternative formulation will produce the correct expression. However the limits $\beta\rightarrow 0$ and $\kappa^2\rightarrow 0$ do commute for the interfacial wave. For the remainder of this section we shall be concerned only with the interfacial wave whose speed is given by (2.13).

For a shallow interface $d \ll 1$, and in R_1 we may put

$$l(z) = l_d\{1+b_dl_d^{-1}(z+d)+O(z+d)^2\}. \quad (2.20)$$

Here $b_d = l_z(-d)$ and is the difference in the slopes of the channel walls at the ends of the interface. Since

$$S_1 = \int_{-d}^{0} l(z)\,dz \quad (2.21)$$

it follows that

$$S_1 = dl_d\{1+\tfrac{1}{2}b_dl_d^{-1}d+O(d^2)\}. \quad (2.22)$$

This expression may be substituted into (2.13) to show that the interfacial wave speed (in the limit $\beta\rightarrow 0$) is approximately given by

$$c_0^2 = 2d\{1+\tfrac{1}{2}b_dl_d^{-1}d-l_dS_2^{-1}d+O(d^2)\}. \quad (2.23)$$

Next let

$$\psi_1 = \tfrac{1}{2}S_1(l_dd)^{-1}z^2+\tfrac{1}{2}(1-S_1(l_dd)^{-1})y^2-\tfrac{1}{2}d^{-1}(S_{1+}-S_{1-})y+C+\hat{\psi}_1, \quad (2.24)$$

where $y = 0$ is chosen to be at the midpoint of the interface $z = -d$ and $S_{1\pm}$ is the area of R_1 which lies in $y \gtrless 0$. If $b_{\pm d} = l_{\pm z}(-d)$, then $S_{1+} - S_{1-}$ is $\frac{1}{2}d^2(b_{+d}+b_{-d})(1+O(d))$. The constant C is so chosen that (2.13) is satisfied. Now $\hat{\psi}_1$ satisfies Laplace's equation in R_1, $\hat{\psi}_{1z}$ is zero on $z = 0, -d$ and $\hat{\psi}_{1y} - l_{\pm z}\hat{\psi}_{1z}$ is $O(d)$ on $y = l_\pm(z)$; also the integral of the last quantity over the sides $y = l_\pm(z)$ is zero. It may then be shown that $\hat{\psi}_1$ is $O(d^2)$ near each end and decays exponentially away from each end within a distance $O(d)$, provided only that $l_{\pm z}$ is $O(1)$ with respect to d. (Convert the Neumann problem for $\hat{\psi}_1$ to a Dirichlet problem for the conjugate function whose boundary values are zero on $z = 0, -d$ and $O(d^2)$ on the sides $y = l_\pm(z)$, whose length is $O(d)$.) Note that (2.24) implies that ψ_{1y} is $O(dl_d)$ (ψ_{1z} is $O(d)$) and that this is $O(1)$ provided that dl_d is $O(1)$. Substituting (2.24) into (2.17) and taking the Boussinesq limit $\beta \to 0$, it follows that

$$\frac{c_1}{c_0^3} = -\frac{1}{4S_2}\int_{z=-d} \psi_2\, dy - \frac{d}{12} + \frac{d}{192}\{b_d^2 - 3(b_{+d}+b_{-d})^2\} + O(d^2). \tag{2.25}$$

Next we shall consider the special case when the channel is rectangular and of total depth h, so that $l_\pm(z) = \pm\frac{1}{2}l_d$ for $0 > z > -h$. Then $l_0 = l_d$, $S_1 = dl_d$ and $S_2 = (h-d)\,l_d$. The interfacial wave speed is given by (in the limit $\beta \to 0$)

$$c_0^2 = 2d(1-dh^{-1}). \tag{2.26}$$

It may be shown that

$$\psi_1 = \tfrac{1}{2}z^2 - \tfrac{1}{6}d^2, \quad \psi_2 = \tfrac{1}{2}(h+z)^2 - \tfrac{1}{6}(h-d)^2 \tag{2.27}$$

and hence

$$c_1/c_0^3 = -\tfrac{1}{12}h. \tag{2.28}$$

When the channel cross-section is a triangle of total depth h, we may put $l(z) = l_0 h^{-1}(h+z)$ for $0 > z > -h$. Hence

$$l_d = l_0 h^{-1}(h-d), \quad b_d = l_0 h^{-1}, \quad S_1 = \tfrac{1}{2}l_0 h^{-1}(2hd - h^2), \quad S_2 = \tfrac{1}{2}l_d(h-d).$$

The interfacial wave speed is given by (in the limit $\beta \to 0$)

$$c_0^2 = 2d(1-dh^{-1})(1-\tfrac{1}{2}dh^{-1}). \tag{2.29}$$

Remarkably, this is independent of the width of the interface [(2.10) shows that, in general, c_0 is a function of β, S_1/l_d, S_2/l_d, and l_0/l_d, but for the case of a triangular cross-section, the last three parameters are functions of d and h alone]. Also the ratio of the speed in a triangular channel to that in a rectangular channel of the same depth is $(1-\frac{1}{2}dh^{-1})^{\frac{1}{2}}$, which varies from 1 when $d = 0$ to 0·7 when $d = h$. To find c_1 we must find ψ_1, which in general must be achieved numerically. However if the cross-section is an isosceles triangle (i.e. $l_+ = -l_-$) then it may be shown that

$$\psi_2 = \tfrac{1}{4}\{y^2 + (h+z)^2\} - \tfrac{1}{8}(h-d)^2(1+\tfrac{1}{12}b_d^2). \tag{2.30}$$

Note that ψ_{2y} is $O(l_d)$ and so, in this instance, l_d must be $O(1)$. Next, from the shallow-interface approximation (2.25) it follows that

$$\frac{c_1}{c_0^3} = -\frac{h}{16}\left(1+\frac{b_d^2}{12}\right) - \frac{d}{48}\left(1-\frac{b_d^2}{2}\right) + O(d^2). \tag{2.31}$$

Finally, we note that the appropriate measure for dispersive effects is $c_1 c_0^{-1}\kappa^2$. In terms of the dimensional wavelength $\bar{\lambda}$, $\kappa = 2\pi\bar{h}\bar{\lambda}^{-1}$, and so the measure for dispersion is $c_1 c_0^{-1}\bar{h}^2\bar{\lambda}^{-2}$. For a shallow interface (2.28) and (2.31) suggest that $c_1 c_0^{-3}$ varies as h, while from (2.23), c_0^2 varies as d; in this case the measure for dispersion is $\bar{d}\bar{h}\bar{\lambda}^{-2}$, where $\bar{d}$ is the dimensional depth of the interface.

3. Nonlinear long waves in a channel of varying cross-section

For weakly nonlinear long waves whose linear dispersion relation has the form (2.6) it is well known that if ϵ is a measure of weak dispersion then ϵ^2 is the appropriate measure of wave amplitude. We shall also allow the channel cross-section to vary slowly in the x direction so we shall allow $l_\pm$ to be functions of x. It was shown by Johnson (1973*a*) that, for surface waves travelling in a rectangular channel of varying depth, the appropriate measure for the slow variation of depth is ϵ^3. It is easily verified that this is also the appropriate measure in the present case, and so we let $l_\pm = l_\pm(z, X)$, where

$$X = \epsilon^3 x. \tag{3.1}$$

The boundary condition (1.6) then becomes

$$\phi_{iy} = l_{\pm z}\phi_{iz} + \epsilon^3 l_{\pm X}\phi_{ix} \quad \text{on} \quad y = l_\pm(z, X), \quad i = 1, 2. \tag{3.2}$$

The cross-sectional areas S_1 and S_2 and the interface lengths l_d and l_0 are now functions of X, so c_0 and c_1 are functions of X given by (2.10) and (2.17) respectively; also ψ_i is a function of y, z and X, the dependence on X being determined parametrically from (2.12).

We anticipate that, to leading order in ϵ, the waves will travel with speed c_0 and hence we introduce the convected co-ordinate

$$\xi = \epsilon^{-2}\int_0^X \{c_0(X')\}^{-1} dX' - \epsilon t. \tag{3.3}$$

Then we seek a solution in which η and ζ are functions of ξ and X and the ϕ_i are functions of y, z, ξ and X. We note that

$$\frac{\partial}{\partial t} \equiv -\epsilon\frac{\partial}{\partial \xi}, \quad \frac{\partial}{\partial x} \equiv \epsilon c_0^{-1}\frac{\partial}{\partial \xi} + \epsilon^3\frac{\partial}{\partial X} \tag{3.4}$$

and put

$$\eta = \epsilon^2(\eta^{(0)} + \epsilon^2\eta^{(1)} + \ldots), \quad \zeta = \epsilon^2(\zeta^{(0)} + \epsilon^2\zeta^{(1)} + \ldots), \quad \phi_i = \epsilon(\phi_i^{(0)} + \epsilon^2\phi_i^{(1)} + \ldots). \tag{3.5}$$

These expressions are then substituted into (1.1) and the boundary conditions (1.4), (1.5) and (3.2). Before proceeding with this calculation we note that (1.1) becomes

$$\phi_{iyy} + \phi_{izz} + \epsilon^2 c_0^{-2}\phi_{i\xi\xi} + 2\epsilon^4 c_0^{-1}\phi_{i\xi X} - \epsilon^4 c_{0x} c_0^{-2}\phi_{i\xi} + \epsilon^6\phi_{ixx} = 0 \quad \text{in} \quad R_i, \quad i = 1, 2. \tag{3.6}$$

At the lowest order in ϵ it may easily be shown that ϕ_i is independent of y and z and hence we put

$$\phi_i^{(0)} = A_i(\xi, X), \quad i = 1, 2. \tag{3.7}$$

These functions A_i are the counterparts of the constants A_i obtained in §2. Comparing (3.7) with (3.4) and (3.5) we see that derivatives with respect to x and t are $O(\epsilon)$, while derivatives with respect to y and z are $O(\epsilon^2)$. As we noted in the introduction, this is the crucial assumption in the theory, rather than the assumption that the horizontal dimension of the channel is of the same order of magnitude as the vertical dimension. Also the boundary conditions (1.4) and (1.5) show that

$$\zeta^{(0)} = \beta A_{1\xi}, \quad \eta^{(0)} = \tfrac{1}{2}(1+\beta)A_{2\xi} - \tfrac{1}{2}(1-\beta)A_{1\xi}. \tag{3.8}$$

At the next order in ϵ, it follows from (3.6) and the boundary conditions (3.2), (1.4) and (1.5) that

$$\left.\begin{aligned} \phi^{(1)}_{iyy} + \phi^{(1)}_{izz} &= -c_0^{-2}A_{i\xi\xi} \quad \text{in} \quad R_i, \quad i = 1,2, \\ \phi^{(1)}_{iy} &= l_{\pm z}\phi^{(1)}_{iz} \quad \text{on} \quad y = l_\pm(z), \quad i = 1,2, \\ \phi^{(1)}_{1z} &= -\zeta^{(0)}_\xi \quad \text{on} \quad z = 0, \\ \phi^{(1)}_{1z} = \phi^{(1)}_{2z} &= -\eta^{(0)}_\xi \quad \text{on} \quad z = -d. \end{aligned}\right\} \tag{3.9}$$

Also the boundary conditions (1.4) and (1.5) show that

$$\left.\begin{aligned} \zeta^{(1)} &= \beta(\phi^{(1)}_{1\xi} - \tfrac{1}{2}c_0^{-2}A^2_{1\xi}) \quad \text{on} \quad z = 0, \\ 2\eta^{(1)} &= (1+\beta)\phi^{(1)}_{2\xi} - (1-\beta)\phi^{(1)}_{1\xi} - \tfrac{1}{2}c_0^{-2}(1+\beta)A^2_{2\xi} + \tfrac{1}{2}c_0^{-2}(1-\beta)A^2_{1\xi} \quad \text{on} \quad z = -d. \end{aligned}\right\} \tag{3.10}$$

The solution of (3.9) is given by

$$\phi^{(1)}_i = -c_0^{-2}A_{i\xi\xi}\psi_i + B_i, \quad i = 1,2, \tag{3.11}$$

where B_i is an undetermined function of ξ and X and ψ_i is defined by (2.12) and (2.13). For (3.11) to be a solution the compatibility condition (2.8) must be satisfied, and this leads to (2.9) and (2.10) as before. At this stage c_0 is a known function of X, and from (2.9) and (3.8), A_1, A_2 and $\xi^{(0)}$ are known in terms of $\eta^{(0)}$.

At second order in ϵ we find that

$$\left.\begin{aligned} \phi^{(2)}_{iyy} + \phi^{(2)}_{izz} &= -c_0^{-2}\phi^{(1)}_{i\xi\xi} - 2c_0^{-1}A_{i\xi X} + c_{0X}c_0^{-2}A_{i\xi} \quad \text{in} \quad R_i, \quad i = 1,2, \\ \phi^{(2)}_{iy} &= l_{\pm z}\phi^{(2)}_{iz} + l_{\pm X}c_0^{-1}A_{i\xi} \quad \text{on} \quad y = l_\pm(z), \quad i = 1,2, \\ \phi^{(2)}_{1z} &= -\zeta^{(1)}_\xi + c_0^{-2}(\zeta^{(0)}A_{1\xi})_\xi + \zeta^{(0)}\phi^{(1)}_{1yy} \quad \text{on} \quad z = 0, \\ \phi^{(2)}_{iz} &= -\eta^{(1)}_\xi + c_0^{-2}(\eta^{(0)}A_{i\xi})_\xi + \eta^{(0)}\phi^{(1)}_{iyy} \quad \text{on} \quad z = -d, \quad i = 1,2. \end{aligned}\right\} \tag{3.12}$$

The solution for $\phi^{(2)}_i$ must satisfy the compatibility condition (2.8); as in §2 this leads to two equations for B_i, and the condition that these equations have a solution for B_i provides the required equation for $\eta^{(0)}$. Examination of (3.10) and the inhomogeneous terms in (3.12) shows that the equation for $\eta^{(0)}$ involves $\eta^{(0)}$, $\eta^{(0)}_X$, $\eta^{(0)}\eta^{(0)}_\xi$ and $\eta^{(0)}_{\xi\xi\xi}$. After considerable manipulation we find that

$$\eta^{(0)}_X + \delta\eta^{(0)}\eta^{(0)}_\xi - c_1c_0^{-4}\eta^{(0)}_{\xi\xi\xi} + \nu\eta^{(0)} = 0, \tag{3.13}$$

where c_1 (a function of X) is defined by (2.17). The presence of c_1 in the coefficient of $\eta^{(0)}_{\xi\xi\xi}$ is to be expected from the linear dispersion relation (2.6). The coefficient δ is given by

$$\delta\left\{1 + \frac{\beta(1-\beta)l_0l_dc_0^4}{2(S_1-\beta c_0^2l_0)^2}\right\} = -\frac{b_d}{2l_dc_0} - \frac{\beta^2(1-\beta)b_0l_d^2c_0^5}{4(S_1-\beta c_0^2l_0)^3} + \frac{3l_d^2c_0}{4}\left\{\frac{(1+\beta)}{S_2^2} - \frac{(1-\beta)S_1}{(S_1-\beta c_0^2l_0)^3}\right\}. \tag{3.14}$$

Here $b_d = l_z(-d, X)$ and $b_0 = l_z(0, X)$. Also the coefficient ν is given by

$$\left.\begin{aligned} \nu &= \gamma^{-1}\gamma_X, \\ \gamma^2 &= c_0 l_d \left\{1 + \frac{\beta(1-\beta)\, l_0 l_d c_0^4}{2(S_1 - \beta c_0^2 l_0)^2}\right\}. \end{aligned}\right\} \tag{3.15}$$

where

Equation (3.13) may be simplified by putting

$$\hat{\eta} = \gamma\eta^{(0)}, \tag{3.16}$$

and then it follows that

$$\hat{\eta}_X + \delta\gamma^{-1}\hat{\eta}\hat{\eta}_\xi - c_1 c_0^{-4}\hat{\eta}_{\xi\xi\xi} = 0. \tag{3.17}$$

Equation (3.17) can now be recognized as a variable-coefficient Korteweg–de Vries equation. The coefficient γ may be interpreted as the appropriate Green's law factor for the present problem. In the linear, non-dispersive limit (3.17) shows that $\hat{\eta}$ is conserved; it may be shown that this result is equivalent to the conservation of energy flux.

First let us note that if the channel has uniform cross-section, so that, in particular, c_0 is a constant, then $\xi = \epsilon(c_0^{-1}x - t)$. If we put $\tau = c_0\xi = \epsilon(x - c_0 t)$ and $T = \epsilon^3 t$, (3.13) then reduces to the more familiar constant-coefficient Korteweg–de Vries equation

$$\eta_T^{(0)} + c_0^2\delta\eta^{(0)}\eta_\tau^{(0)} - c_1\eta_{\tau\tau\tau}^{(0)} = 0, \tag{3.18}$$

at least to within an error of $O(\epsilon^2)$.

Next let us consider the Boussinesq limit $\beta \to 0$. The phase speed c_0 is then given by (2.18) for the interfacial wave and (2.19) for the surface wave. In the same limit, for the interfacial wave

$$\left.\begin{aligned} c_0\delta &= -\frac{b_d}{2l_d} + \frac{3l_d(S_1 - S_2)}{2S_1 S_2} + O(\beta), \\ \gamma &= (c_0 l_d)^{\frac{1}{2}} + O(\beta). \end{aligned}\right\} \tag{3.19}$$

Since, for most applications, $S_2 > S_1$ and b_d is positive, (3.19) shows that δ will usually be negative. For the surface wave,

$$\left.\begin{aligned} \frac{c_0 l_0 S_2 \delta}{l_d(S_1 + S_2)} &= -\frac{b_0}{2l_0} + \frac{3l_0}{2(S_1 + S_2)} + O(\beta), \\ \frac{l_0 S_2 \gamma}{l_d(S_1 + S_2)} &= \frac{(c_0 l_0)^{\frac{1}{2}}}{2\beta} + \{1 + O(\beta)\}. \end{aligned}\right\} \tag{3.20}$$

In the limit $\beta \to 0$

$$\eta^{(0)} = \zeta^{(0)}\frac{l_0 S_2}{l_d(S_1 + S_2)}\{1 + O(\beta)\} \tag{3.21}$$

and it may be shown that (3.17) becomes

$$\left.\begin{aligned} \zeta_X + \delta'\gamma'^{-1}\zeta\zeta_\xi - c_1 c_0^{-4}\zeta_{\xi\xi\xi} &= 0, \\ \zeta &= \gamma'\zeta^{(0)}, \\ \gamma' &= (c_0 l_0/2\beta)^{\frac{1}{2}}\{1 + O(\beta)\}, \\ c_0\delta' &= -\frac{b_0}{2l_0} + \frac{3l_0}{2(S_1 + S_2)} + O(\beta). \end{aligned}\right\} \tag{3.22}$$

where

In the limit $\beta \to 0$, (3.22) is the variable-coefficient Korteweg–de Vries equation for surface waves in a channel of varying cross-section. If the channel has uniform cross-section then (3.22) agrees with the Korteweg–de Vries equation derived by Peregrine (1968), for surface waves in a channel of uniform cross-section. Equation (3.22) also generalizes the result obtained by Johnson (1973*a*) for surface waves in a rectangular channel of varying depth. In this case $\beta c_0^2 = h$, $c_0 \delta' = 3/2h$ and $c_1 c_0^{-3} = -\frac{1}{12}h$, and our results agree with those obtained by Johnson (*loc. cit.*).

In the shallow-interface approximation ($d \ll 1$), c_0 is given by (2.23) and c_1 is given by (2.25). For the interfacial wave, (3.19) becomes

$$\left.\begin{aligned} \delta &= -\frac{3}{(2d)^{\frac{3}{2}}}\left\{1 - \frac{l_d d}{2S_2} - \frac{5b_d d}{12 l_d} + O(d^2, \beta)\right\}, \\ \gamma &= l_d^{\frac{1}{2}}\,(2d)^{\frac{1}{4}}\left\{1 - \frac{l_d d}{4S_2} + \frac{b_d d}{8 l_d} + O(d^2, \beta)\right\}. \end{aligned}\right\} \qquad (3.23)$$

Since d does not vary with X, we see from (2.23) and (3.23) that c_0 and δ are largely unaffected by the variable cross-section when the interface is shallow, while γ varies principally as $l_d^{\frac{1}{2}}$. The coefficient c_1 is given by (2.25) and may be expected to depend on the cross-section S_2 in a complicated manner.

Next we shall consider the special case when the channel is a rectangle of total depth h. Then c_0 and c_1 are given by (2.26) and (2.28), while from (3.19)

$$c_0^3 \delta = -3(1 - 2d/h) + O(\beta). \qquad (3.24)$$

This result agrees with that obtained by Keulegan (1953; see also Long 1956; Lee & Beardsley 1974). Finally, consider the case when the channel is a triangle of total depth h and $l(z) = l_0(h+z)/h$ for $0 > z > -h$. Then c_0 and c_1 are given by (2.29) and (2.31) while from (3.19) it may be shown that

$$2c_0^3 \delta = (-6 + 22d/h - 11d^2/h^2)\,(1 + O(\beta)). \qquad (3.25)$$

Thus δ is negative (positive) when $d/h < 0{\cdot}33$ ($> 0{\cdot}33$). This result may be compared with the corresponding result (3.24) for a rectangle, where δ is negative (positive) when $d/h < 0{\cdot}50$ ($> 0{\cdot}50$). Like c_0, δ is independent of the width of the interface.

4. Discussion

In this section we shall consider the solutions of (3.13) or equivalently (3.17), which we shall display again for convenience:

$$\hat{\eta}_X + \delta\gamma^{-1}\hat{\eta}\hat{\eta}_\xi - c_1 c_0^{-4}\hat{\eta}_{\xi\xi\xi} = 0, \quad \text{where} \quad \hat{\eta} = \gamma\eta^{(0)}. \qquad (4.1)$$

Here $\eta^{(0)}$ is the height of the interface, the 'Green's law' factor γ is defined by (3.15), the nonlinear coefficient δ is defined by (3.14) and the dispersion coefficient c_1 is defined by (2.17). Also we recall that ξ is the convected co-ordinate

$$\epsilon^{-2}\int_0^X \{c_0(X')\}^{-1}\,dX' - \epsilon t, \quad X = \epsilon^3 x,$$

and c_0 is the linear long-wave phase speed (2.10). On the basis of the shallow-interface and Boussinesq approximation (3.23), we shall assume that the coefficient δ is negative.

Also we note that c_0 is positive and c_1 is negative. Equation (4.1) is to be solved with an 'initial' condition that specifies $\eta^{(0)}$ as a function of ξ at some particular fixed value of X; from (3.3) this is equivalent to specifying $\eta^{(0)}$ as a function of time (at some fixed value of X).

4.1. *Nonlinear steepening*

Observations of long internal waves show that the generation of an internal surge is followed by a phase in which nonlinear steepening is the dominant process, which is followed in turn by the formation of shorter period waves due to the interplay between dispersion and nonlinearity (see Lee & Beardsley 1974; Farmer 1978). In this subsection we shall ignore dispersive effects, and so ignore the term whose coefficient is c_1 in (3.17). The equation to be considered is thus

$$\hat{\eta}_X + \delta\gamma^{-1}\hat{\eta}\hat{\eta}_\xi = 0, \quad \hat{\eta} = \gamma\eta^{(0)}. \tag{4.2}$$

The general solution of (4.2) is

$$\left.\begin{aligned} \eta^{(0)} &= \sigma^{-1}f(\xi + \sigma\eta^{(0)}S), \\ \text{where} \quad S &= -\int_0^X \delta\sigma^{-1}\,dX, \quad \sigma = \gamma\gamma_0^{-1}. \end{aligned}\right\} \tag{4.3}$$

Here γ_0 (a constant) is the value of γ at $X = 0$, and $f(\xi)$ is the specified initial value of $\eta^{(0)}$ at $X = 0$. The solution (4.3) remains valid provided that

$$1 \neq Sf'(\xi + \sigma\eta^{(0)}S). \tag{4.4}$$

Since δ is negative, S is positive and (4.4) cannot be violated if f' is negative; if f' is positive then the first value of S at which (4.4) is violated defines a breaking distance. In general (4.3) shows that waves of elevation break backwards as they propagate, while waves of depression break forwards. These results agree with those of Long (1972), who considered the special case when the channel has a constant rectangular cross-section. Of course, as the surge begins to steepen, dispersive effects become important and the last term in (4.2) must be retained; this will be discussed in §4.2. Meanwhile we shall continue to discuss (4.2).

Further progress depends on specifying the coefficients δ and γ. In the Boussinesq limit ($\beta \to 0$) and the shallow-interface approximation ($d \ll 1$), δ and γ are given by (3.23); since d does not vary with X, δ remains effectively constant while γ varies as $l_d^{\frac{1}{2}}$. Farmer (1978) has observed internal surges in Lake Babine, which narrows by a factor of two in the direction of the surge propagation. If γ decreases with increasing X, then it is apparent from (4.2) and (4.3) that the steepening process is enhanced both by the Green's Law factor and by the increased value of S for a given X. To be specific let us assume that l_d varies linearly with X, so that

$$\sigma = (1 - qX)^{\frac{1}{2}}, \tag{4.5}$$

where q is a constant. Then, on substituting into (4.2), it follows that

$$-\delta X = S(1 + \tfrac{1}{4}q\delta^{-1}S). \tag{4.6}$$

For instance, for that value of X at which l_d has been reduced by a factor of two, the fractional increase in the value of S relative to $-\delta X$ (the value of S if l_d is constant, i.e. $q = 0$ or $\sigma = 1$) is approximately $\frac{1}{7}$. To gauge the effect on the wave amplitude, let

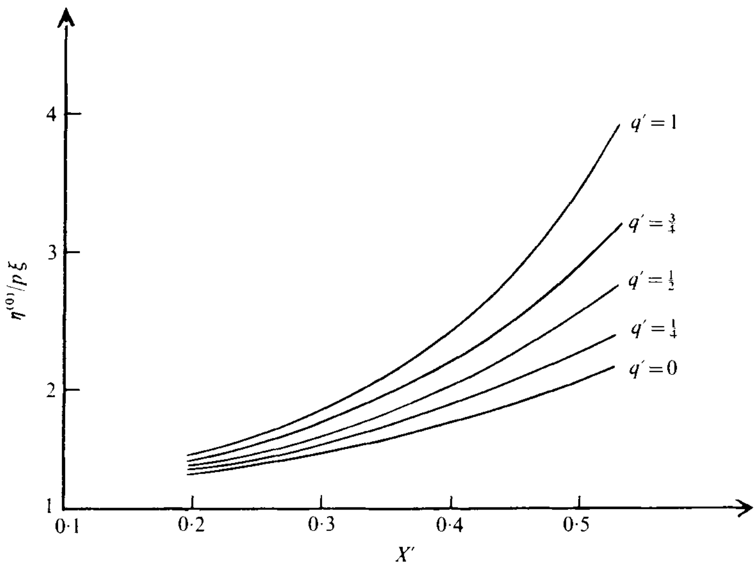

FIGURE 2. Plot of $\eta^{(0)}/p\xi$ against X' [see (4.9)] for $q' = 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1$.

us suppose that $f(\xi)$ contains a segment of constant positive slope p and that outside this segment $f'(\xi)$ is negative. Specifically we put

$$f(\xi) = p\xi \quad \text{for} \quad |\xi| < \xi_1. \tag{4.7}$$

The solution determined from (4.3) is

$$\sigma\eta^{(0)} = p\xi(1-pS)^{-1} \quad \text{for} \quad |\xi| < \xi_1(1-pS). \tag{4.8}$$

The solution is valid for $pS < 1$, and fails at $pS = 1$, which defines the breaking distance. The solution in the region $|\xi| > \xi_1(1-pS)$ need not concern us, as there is no steepening in this region. We wish to compare this solution with the corresponding solution when l_d is a constant ($q = 0$ or $\sigma = 1$). From (4.6) and (4.8), $\eta^{(0)}(p\xi)^{-1}$ is a function of $X' = p|\delta|X$, with a parametric dependence on $q' = q(p|\delta|)^{-1}$:

$$\frac{\eta^{(0)}}{p\xi} = \sigma^{-1}\left\{1-\frac{2}{q'}(1-\sigma)\right\}^{-1}, \tag{4.9}$$

where

$$\sigma = (1-q'X')^{\frac{1}{2}}.$$

Figure 2 displays this function for various values of q'. When $q' = 0$, $\eta^{(0)}$ has doubled in value when $X' = 0{\cdot}5$; by contrast, when $q' = 1$ (i.e. l_d has been reduced by a factor of two), $\eta^{(0)}$ has increased in value by a factor of 3·4. In summary, a decrease in the width of the channel can be as effective as nonlinear processes in increasing the amplitude of the surge, and this may be relevant to the large amplitude surges observed in Lake Babine. However, in comparing our theory with observations, it should be noted that the theory ignores Coriolis forces, which may be significant in the development of the surge, which takes place on a time scale of a day, rather than an hour.

4.2. *Soliton formation*

When the internal surge approaches the breaking distance, dispersive effects become important and the equation to be solved is than (4.1), with an initial condition that specifies $\eta^{(0)}$ at some appropriate value of X; the initial $\eta^{(0)}$ is given by (4.3) and one of the principal effects of allowing the cross-section to vary is to alter this initial condition. If the Korteweg–de Vries equation (4.1) has constant coefficients, then there is a solitary-wave solution, or soliton:

$$\left.\begin{aligned} \eta^{(0)} &= a\,\mathrm{sech}^2\{k(\xi - V^{-1}X)\}, \\ \text{where}\qquad V &= 3(\delta a)^{-1}, \quad 4k^2 = -a\delta c_0^{\frac{4}{3}}(3c_1)^{-1}. \end{aligned}\right\} \tag{4.10}$$

Since c_1 and δ are negative, the amplitude a is also negative and this is a wave of depression travelling in the $+X$ direction. Its speed is $c_0(1+\epsilon^2 c_0 V^{-1})$ and its e-folding width is $2c_0(\epsilon k)^{-1}$. It is now well known that the constant-coefficient Korteweg–de Vries equation also has N-soliton solutions (see Whitham 1974, chap. 17) and that an initially smooth wave profile will evolve into a train of solitons. If $f(\xi)$ is the initial condition, then an asymptotic expression for the number N of solitons produced is

$$N \sim \frac{1}{\pi}\left(\frac{\delta c_0^4}{6c_1}\right)^{\frac{1}{2}} \int_{-\infty}^{\infty} |f(\xi)|^{\frac{1}{2}}\, d\xi. \tag{4.11}$$

Using the Boussinesq limit ($\beta \to 0$) and the shallow-interface approximation, we can assume that δ and c_0 do not vary with X. Then (4.10) shows that the solitary-wave speed (for a given amplitude) is unaffected by a change in the cross-section, but that the width is affected, being decreased by a decrease in c_1. Examining (4.11), we see that the number N of solitons produced is increased by a decrease in c_1, and is also increased by an increase in the initial condition, i.e.

$$\int_{-\infty}^{\infty} |f(\xi)|^{\frac{1}{2}}\, d\xi$$

is increased.

Hunkins & Fliegel (1973) have observed the formation of a train of short period internal waves in Seneca Lake and Farmer (1978) has made similar observations in Lake Babine. The latter are particularly interesting as the formation of a train of internal waves was most prominent at a point in the lake where the cross-section had narrowed by a factor of two from the region where the internal surge was generated; also the maximum depth decreases by a factor of three. If we approximate the cross-section by a triangle and use (2.26) to estimate c_1, then c_1 decreases by approximately 25 % from the generation region, thus favouring the formation of solitons. However, the major factor affecting the formation of solitons in Lake Babine is likely to be the decrease in the width l_d and the corresponding decrease in γ, which causes an increase in the nonlinear steepening as described in §4.1. If $\bar{\lambda}$ is the dimensional width of a solitary wave, $\bar{h}$ is the dimensional depth and $\bar{a}$ is the dimensional amplitude, then (4.10) shows that

$$\frac{\bar{a}\bar{\lambda}^2}{\bar{h}^3} = \frac{-4c_0^2 a}{k^2} = \frac{36c_1}{\delta c_0^2}. \tag{4.12}$$

For a shallow interface $c_1 c_0^{-3}$ varies as h [cf. (2.28) and (2.31)], c_0^2 varies as d [cf. (2.23)], while δ varies as $d^{-\frac{3}{2}}$ [cf. (3.23)]. The right-hand side of (4.12) then varies as d^2. Hence we may deduce from (4.12) that nonlinear effects are balancing dispersive effects when $\bar{a}\,\bar{\lambda}^2$ is comparable with $\bar{h}\bar{d}^2$, where $\bar{d}$ is the dimensional depth of the interface; since $\bar{d}\bar{h}\bar{\lambda}^{-2}$ is the appropriate measure for dispersion when the interface is shallow, this implies that the appropriate measure for nonlinear effects is $\bar{a}\,\bar{d}^{-1}$. Farmer (1978) has observed internal waves of amplitude 10 m (in a depth of approximately 100 m), phase speed 0·2 m s^{-1} and wavelength approximately 460 m. Using (2.31) we estimate that $c_1 c_0^{-3}$ is approximately $-0{\cdot}77$ at the point where the waves are observed (b_d is approximately 15). Also, using (2.29) to estimate c_0 and (3.25) to estimate δ, with $dh^{-1} = 0{\cdot}2$, we find that (4.12) gives a value for $\bar{\lambda}$ of 550 m. That this is an overestimate is probably due to an overestimate of c_1; if the lake is modelled by a rectangular channel then $c_1 c_0^{-3}$ is $-\frac{1}{12}$; in general, c_1 is very sensitive to the shape of the cross-section. By contrast, c_0 and δ depend mainly on d and are fairly insensitive to the shape of the cross-section. Using (2.29) with $dh^{-1} = 0{\cdot}2$, we find that the theoretical dimensional phase speed is 0·24 m s^{-1} (for the observed temperature difference across the interface in Lake Babine $\beta = 0{\cdot}2 \times 10^{-3}$).

4.3. *Slowly varying solitary wave*

When the coefficients of the Korteweg–de Vries equation (4.1) are functions of X, there is no simple analytic solution. However, when the variation in the channel cross-section is $O(\sigma\epsilon^3)$, where $\sigma \ll 1$, then there is an asymptotic solution which describes a slowly varying solitary wave. The procedure for constructing this solution has been described by Grimshaw (1970) and Johnson (1973*b*) for surface solitary waves travelling in a rectangular channel of varying depth, so we shall give only a brief description here.

We shall suppose that c_0, c_1, γ and δ are functions of $\hat{X} = \mu X$ and we shall then seek a solution of (3.17) in which $\eta^{(0)}$ depends on $\hat{X}$ and θ, where

$$\left.\begin{aligned} \eta^{(0)} &= a\,\mathrm{sech}^2 k\theta + \mu\eta_1^{(0)} + O(\mu^2),\\ \theta &= \xi - \int_0^{\hat{X}} (\mu V)^{-1}\, d\hat{X}. \end{aligned}\right\} \tag{4.13}$$

Here a, k and V are related by the formulae in (4.10), and are functions of $\hat{X}$. Substituting (4.13) into (3.17) shows that, from the term of order one in μ,

$$-\gamma V^{-1}\eta_{1\theta}^{(0)} + \gamma\delta\{a\,\mathrm{sech}^2 k\theta\,\eta_1^{(0)}\}_\theta - \gamma c_1 c_0^{-4}\eta_{1\theta\theta\theta}^{(0)} + \partial(\gamma a\,\mathrm{sech}^2 k\theta)/\partial\hat{X} = 0. \tag{4.14}$$

The necessary and sufficient condition that this equation for $\eta_1^{(0)}$ should have a solution which is bounded as $\theta \to \pm\infty$ is

$$\frac{\partial}{\partial\hat{X}}\int_{-\infty}^{\infty} (\gamma a\,\mathrm{sech}^2 k\theta)^2\, d\theta = 0. \tag{4.15}$$

On integrating and using (4.10), this condition becomes

$$c_1 c_0^{-4}\gamma^4\delta^{-1}a^3 = \text{constant}. \tag{4.16}$$

This equation determines the variation of the amplitude a as $\hat{X}$ varies. In the Boussinesq limit ($\beta \to 0$) and the shallow-interface approximation, c_0 [see (2.23)] and

δ [see (3.23)] do not vary with $\hat{X}$, while γ [see (3.23)] varies as $l_d^{\frac{1}{2}}$; hence a varies as $l_d^{-\frac{2}{3}} c_1^{-\frac{1}{3}}$. In a rectangular channel c_1 [see (2.28)] varies as h, while in a triangular channel c_1 [see (2.31)] varies as $h(1+\frac{1}{12}b_d^2)$.

The result (4.16) may also be applied to a surface solitary wave. If a' is the amplitude of such a wave then

$$c_1 c_0^{-4} \gamma'^4 \delta'^{-1} a'^3 = \text{constant}, \tag{4.17}$$

where γ' and δ' are defined in (3.22). This result follows most directly by applying the above method to (3.22) directly. For a rectangular channel c_0 varies as $h^{\frac{1}{2}}$, c_1 varies as $h^{\frac{3}{2}}$, δ' varies as $h^{-\frac{3}{2}}$ and γ' varies as $l_0^{\frac{1}{2}} h^{\frac{1}{4}}$; it follows from (4.17) that a' then varies as $h^{-1} l_0^{-\frac{2}{3}}$, which agrees with the result obtained by Miles (1977), who considered surface solitary waves in a slowly varying channel (see also Grimshaw (1970) and Johnson (1973*b*), who considered only the effect of changes in the depth h).

I am grateful to Dr D. M. Farmer for showing me a draft of his manuscript prior to publication.

Appendix. Alternative expression for c_1

In this appendix we shall derive an alternative expression to (2.17) for c_1 and hence show that $c_0 c_1$ is negative. First let us consider the integral identity

$$\iint_R \{u(u_{yy}+u_{zz})+u_y^2+u_z^2\}\,dy\,dz = \oint_{\partial R} u(u_y\,dz-u_z\,dy), \tag{A 1}$$

where u is any function (with continuous second derivatives in R). Then apply (A 1) to $\hat{\phi}_i$ in R_i; using (2.5) it follows that

$$\left.\begin{aligned} \kappa^2 I + Q &= \int_{z=0} \beta(1-\beta)\kappa^2 c^2 \hat{\phi}_1^2\,dy + \tfrac{1}{2}\int_{z=-d} \kappa^2 c^2 \{(1-\beta)\hat{\phi}_1 - (1+\beta)\hat{\phi}_2\}^2\,dy,\\ \text{where}\quad I &= (1-\beta)\iint_{R_1} \hat{\phi}_1^2\,dy\,dz + (1+\beta)\iint_{R_2} \hat{\phi}_2^2\,dy\,dz,\\ Q &= (1-\beta)\iint_{R_1} (\hat{\phi}_{1y}^2+\hat{\phi}_{1z}^2)\,dy\,dz + (1+\beta)\iint_{R_2} (\hat{\phi}_{2y}^2+\hat{\phi}_{2z}^2)\,dy\,dz. \end{aligned}\right\} \tag{A 2}$$

If we now differentiate (A 2) with respect to κ, integrate by parts where necessary and use (2.5) again, it may be shown that

$$(dc^2/d\kappa^2)(\kappa^4 I + \kappa^2 Q) = -c^2 Q. \tag{A 3}$$

Since I and Q are positive, it follows immediately that $dc^2/d\kappa^2$ is negative, and comparison with (2.6) shows that $c_0 c_1$ is negative. Further, on substituting (2.6) and (2.11) into (A 3) it may be shown that

$$\left.\begin{aligned} 2c_1/c_0 &= -Q_0/I_0,\\ \text{where}\quad I_0 &= (1-\beta)A_1^2 S_1 + (1+\beta)A_2^2 S_2,\\ Q_0 &= (1-\beta)A_1^2\iint_{R_1} (\psi_{1y}^2+\psi_{1z}^2)\,dy\,dz + (1+\beta)A_2^2\iint_{R_2} (\psi_{2y}^2+\psi_{2z}^2)\,dy\,dz. \end{aligned}\right\} \tag{A 4}$$

Next, using (A 1) and (2.12), it follows that

$$\left.\begin{aligned}\iint_{R_1}(\psi_{1y}^2+\psi_{1z}^2)\,dy\,dz &= \beta c_0^2\int_{z=0}\psi_1\,dy+\frac{(S_1-\beta c_0^2 l_0)}{l_d}\int_{z=-d}\psi_1\,dy,\\ \iint_{R_2}(\psi_{2y}^2+\psi_{2z}^2)\,dy\,dz &= \frac{S_2}{l_d}\int_{z=0}\psi_2\,dy.\end{aligned}\right\} \quad (A\ 5)$$

On substituting (A 5) into (A 4), it is easily shown that (A 4) agrees with (2.17)

REFERENCES

FARMER, D. M. 1978 Observations of long nonlinear internal waves in a lake. *J. Phys. Ocean.* **8** (to appear).

GRIMSHAW, R. 1970 The solitary wave in water of variable depth. *J. Fluid Mech.* **42**, 639–656.

HALPERN, D. 1971 Observations of short-period internal waves in Massachusetts Bay. *J. Mar. Res.* **29**, 116–132.

HUNKINS, K. & FLIEGEL, M. 1973 Internal undular surges in Seneca Lake: a natural occurrence of solitons. *J. Geophys. Res.* **78**, 539–548.

JOHNSON, R. S. 1973*a* On the development of a solitary wave moving over an uneven bottom. *Proc. Camb. Phil. Soc.* **73**, 183–203.

JOHNSON, R. S. 1973*b* On an asymptotic solution of the Korteweg–de Vries equation with slowly varying coefficients. *J. Fluid Mech.* **60**, 813–824.

KEULEGAN, G. H. 1953 Characteristics of internal solitary waves. *J. Res. Nat. Bur. Stand.* **51**, 133–140.

LAMB, H. 1932 *Hydrodynamics*. Cambridge University Press.

LEE, C.-V. & BEARDSLEY, R. C. 1974 The generation of long nonlinear internal waves in a weakly stratified shear flow. *J. Geophys. Res.* **79**, 453–462.

LONG, R. R. 1956 Solitary waves in one- and two-fluid systems. *Tellus* **8**, 460–471.

LONG, R. R. 1972 The steepening of long internal waves. *Tellus* **24**, 88–99.

MILES, J. W. 1977 Note on a solitary wave in a slowly varying channel. *J. Fluid Mech.* **80**, 149–152.

PEREGRINE, D. H. 1968 Long waves in a uniform channel of arbitrary cross-section. *J. Fluid Mech.* **32**, 353–365.

SHEN, M. C. 1968 Long waves in a stratified fluid over a channel of arbitrary cross-section. *Phys. Fluids* **11**, 1853–1862.

SHEN, M. C. & KELLER, J. B. 1973 Ray method for nonlinear wave propagation in a rotating fluid of variable depth. *Phys. Fluids* **16**, 1565–1572.

THORPE, S. A. 1971 Asymmetry of the internal seiche in Loch Ness. *Nature* **231**, 306–308.

THORPE, S. A. 1974 Near-resonant forcing in a shallow two-layer fluid: a model for the internal surge in Loch Ness. *J. Fluid Mech.* **69**, 509–527.

THORPE, S. A., HALL, A. & CROFTS, I. 1972 The internal surge in Loch Ness. *Nature* **237**, 96–98.

WHITHAM, G. B. 1974 *Linear and Nonlinear Waves*. Academic Press.

WINANT, C. D. 1974 Internal surges in coastal waters. *J. Geophys. Res.* **79**, 4523–4526.

ZIEGENBEIN, J. 1969 Short internal waves in the Strait of Gibraltar. *Deep-Sea Res.* **16**, 479–487.